Computing Total Dominator Chromatic Numbers of Some Graphs

Taiyo S. Terada[1]
Portland State University, Portland, OR 97207, USA

Under the direction of Dr. John Caughman
with second reader Dr. Derek Garton

Mathematical literature and problems project
in partial fulfillment of requirements for the
Masters of Science in Mathematics.

Abstract

A *total dominator* coloring of a graph is a proper coloring where, for every vertex v, there is a color c such that v is adjacent to all the vertices of color c. In this case, we say the vertex v *totally dominates* the color c. The minimum number of colors for which a total dominator coloring exists is called the *total dominator chromatic number* of a graph. In this literature project we will discuss total dominator chromatic numbers of paths and cycles, along with a few other graphs.

1. Introduction

Domination in graphs has long been of interest in graph theory. An early example of domination concerned the question of determining how many chess pieces of a certain type (say, knights) are needed so that each location on the board is under attack. Our own initial interest in total dominator colorings was peaked by some apparent inconsistencies in the literature, and a desire to sort them out. There are numerous claims in the literature involving the total dominator chromatic numbers for cycles and paths, and even some confusion about the origins of the definitions of total dominator colorings and total dominator chromatic number. In [5], Vikayalekshmi stated the definition at least as early as 2012, and in [3], Kazemi appears to reintroduce the same concept in 2014. The definitions and results may be independent, as far as we can tell from our research. However, the exposition in [3] appears to contain a number of erroneous claims and proofs - counter-examples to which can be found in [2], [4], and [6].

In this paper, our aim is simply to work through the various results appearing in [2], [4], and [6]; in doing so, we will present formulas for the total dominator chromatic number of cycles and paths. Although we found number of small typos and fixable mistakes, the articles by Vijayalekshmi, appear to have the earliest and most correct proofs and results about cycles and paths. Henning also has an essentially correct proof for paths in [2], although there appears to be an incorrect value in one of the base cases when $n = 18$. (The other implied values turn out to be correct, but the minimality of the given coloring might not be guaranteed.)

2. Basic Definitions and Lemmas

Throughout this discussion $G = (V, E)$ will denote an undirected simple graph with vertex set V, edge set E, and no isolated vertices. We will use mostly standard definitions and notation for

[1]Email: taiyo2@pdx.edu

graphs, consistent with [1].

Definition 2.1. For any positive integer n, the *path* P_n is the graph with vertex set $[n]$ and edge set $E = \{\{i,j\} \subseteq [n] : |i-j| = 1\}$. For $n \geq 3$, the *cycle* C_n is the graph with vertex set $[n]$ and edge set $E = \{\{i,j\} \subseteq [n] : i \equiv j+1 \pmod{n}\}$.

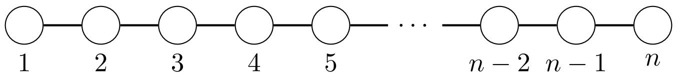


Figure 1: The path P_n.

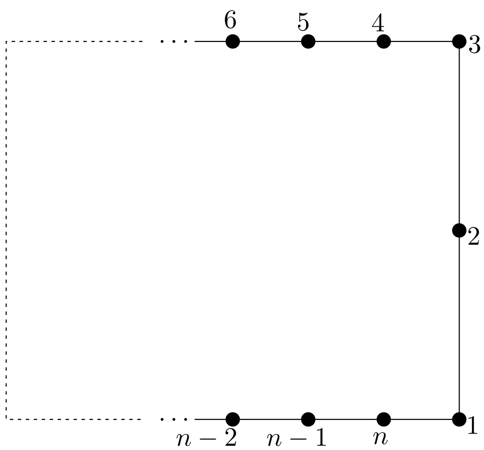


Figure 2: The cycle C_n

Definition 2.2. The *neighbor* set $N(v)$ of a vertex $v \in V$ is the set of vertices in $x \in V$ that are adjacent to v. We will also write $x \sim v$ when x and v are adjacent, i.e. when $\{x,v\} \in E$.

Figure 3: The vertices in $N(v)$ are shown in gray, and $|N(v)| = 8$.

Definition 2.3. A *coloring* of a graph is an assignment of colors to the vertices. A coloring is *proper* if adjacent vertices are assigned different colors. The *chromatic number* $\chi(G)$ is the minimum number of colors in any proper coloring of a graph G. Given a coloring we say a vertex v *totally dominates* a color c if all the vertices of color c are contained in $N(v)$. We say it totally dominates a vertex if it totally dominates the color of that vertex.

Definition 2.4. A *total dominating set*, or *td-set*, is a subset S of the vertex set V such that every vertex in V is adjacent to some vertex in S. The *total domination number* $\gamma_{\text{td}}(G)$ is the minimum cardinality of all total dominating sets of a graph G.

Definition 2.5. A *total dominator coloring*, or td- *td-coloring* is a proper coloring of the vertex set V such that for every vertex $v \in V$, the neighbor set $N(v)$ contains some color class as a subset. Equivalently, for every $v \in V$ there is some color c such that v is adjacent to every vertex of color c. The *total dominator chromatic* number $\chi_{\text{td}}(G)$ is the minimum number of colors in any total dominator coloring of a graph G.

Note that any graph with isolated vertices cannot have a total dominating set, nor can it have a total dominator coloring, hence we make a global assumption that our graphs have no isolated vertices. The following observation appears in Henning [2, pg. 956] without proof.

Lemma 2.6. *[2, pg. 956] For $n \geq 2$*

$$\gamma_{\text{td}}(C_n) = \gamma_{\text{td}}(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4, \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod 4, \\ \frac{n+1}{2} & \text{otherwise.} \end{cases}$$

Alternatively, this can be written as

$$\gamma_{\text{td}}(C_n) = \gamma_{\text{td}}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

Proof. We will show the proof for paths, and the proof for cycles follows by a similar argument. It can be checked by brute force that

$$\gamma_{\text{td}}(C_n) = \gamma_{\text{td}}(P_n) = \begin{cases} 2 & \text{if } n = 2, 3, 4 \\ 3 & \text{if } n = 5, \end{cases}$$

and minimum td-sets are shown below in gray.

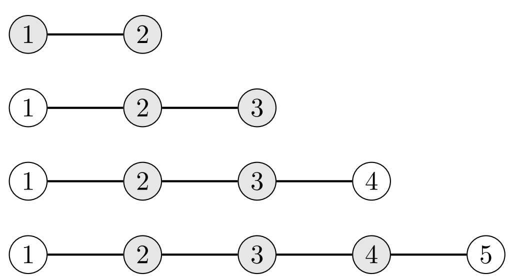


Figure 4: Base cases for P_n

We can construct total dominating sets that meet the desired bound by taking all vertices congruent to 2 and 3 $\pmod 4$, and less than $n - r$, where $n \equiv r \pmod 4$ and $r \in \{2, 3, 4, 5\}$, then adding to our set the gray vertices for $i > n - r$ as shown in the following figure.

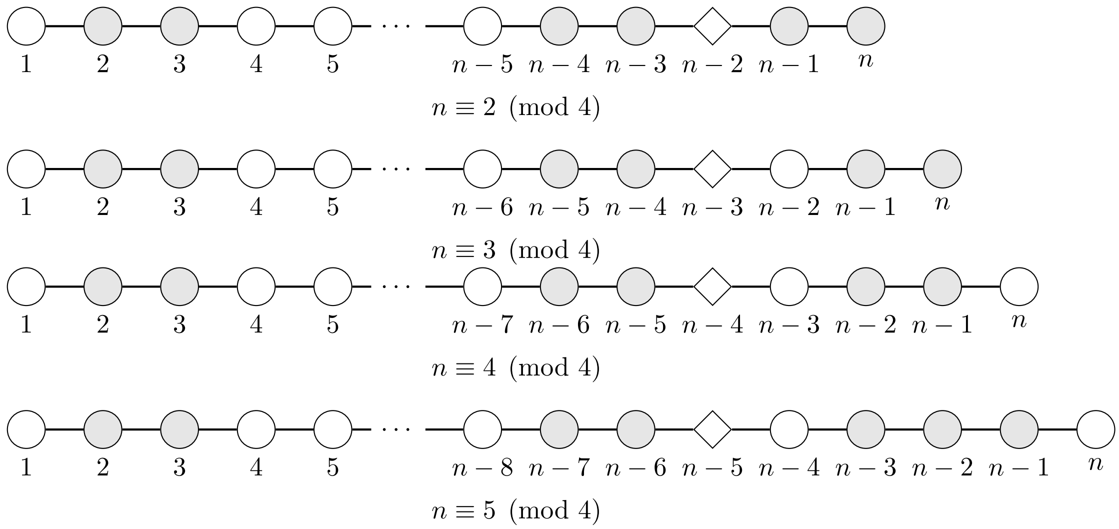


Figure 5: Inductive steps. The diamond shaped vertex indicates the last vertex congruent to 0 (mod 4) that follows the standard pattern described above.

(Notice, the pattern of the total dominating set after the diamond vertex matches our base cases.)

To see that these dominating sets are minimum size, we proceed by induction. Suppose $\gamma_{\text{td}}(P_n) = m$. It suffices to show that $\gamma_{\text{td}}(P_{n+4}) = m + 2$. By the above construction, we have that $\gamma_{\text{td}}(P_{n+4}) \leq m + 2$. Towards a contradiction, suppose we have a td-set of P_{n+4} of size at most $m + 1$. Looking at the vertices $n, n+1, n+2, n+3, n+4$, we know that $n+4$ can only be totally dominated by $n+3$, and $n+3$ must be totally dominated by either $n+2$ or $n+4$.

If n is not totally dominated by $n+1$, then the induced subgraph on the vertices $[n]$ is totally dominated by the set $S \cap [n]$, which has size at most $m + 1 - 2 = m - 1$, contradicting that $\gamma_{\text{td}}(P_n) = m$.
If n is totally dominated by $n+1$, the set $(S \cap [n]) \cup \{n-1\}$ is a td-set of the subgraph induced by the set $[n]$. Since $n+1$, $n+3$, and at least one of $n+2$ or $n+4$ are in S, this means

$$|(S \cap [n]) \cup \{n-1\}| \leq m + 1 - 3 + 1 = m - 1,$$

again contradicting that $\gamma_{\text{td}}(P_n) = m$. □

We now present a result first presented by Vijayalekshmi in [5]. Henning in [2, pg. 958] presents a proof, and attributes a similar statement to Kazemi [3].

Lemma 2.7. *[2, pg. 958] For any graph G with no isolated vertices,*

$$\max\{\chi(G), \gamma_{\text{td}}(G)\} \leq \chi_{\text{td}}(G) \leq \chi(G) + \gamma_{\text{td}}(G).$$

Proof. Take any total dominator coloring of G. This is a proper coloring, and if we take one vertex from each color class, that will be a total dominating set. This gives us the first inequality.

Take a proper coloring of G with color set C, where $|C| = \chi(G)$. Then take a minimal total dominating set S, hence of size $\gamma_{\text{td}}(G)$. Color each vertex in S with its own new unique color not in C. This is a td-coloring with exactly $\chi(G) + \gamma_{\text{td}}(G)$ colors, giving us our second inequality.

□

Definition 2.8. The *wheel* graph with $n+1$ vertices, notated by W_n, is the graph with vertex set $[n] \cup \{c\}$, and edge set $\{\{i,j\} \subseteq [n] : i+1 \equiv j \pmod{n}\} \cup \{\{i,c\} : i \in [n]\}$.

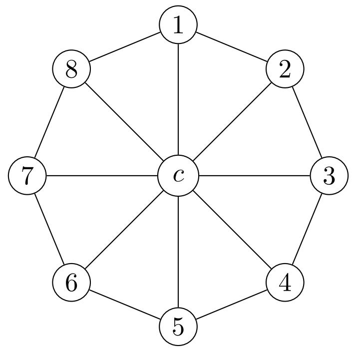


Figure 6: Wheel graph W_8.

Corollary 2.9. *[3, pg.62] For $n \geq 2$,*

$$\chi_{\mathrm{td}}(W_n) = \begin{cases} 3 & \textit{if } n \textit{ is even,} \\ 4 & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. It is easy to see that $\chi(G)$ is 3 when n is even and 4 when n is odd. So by Lemma 2.7, if n is even, we have $\max\{3,2\} \leq \chi_{\mathrm{td}}(W_n) \leq 2+3$. If n is odd, we have $\max\{4,2\} \leq \chi_{\mathrm{td}}(W_n) \leq 2+4$. However, since vertex c is adjacent to all vertices, it has a unique color in any proper coloring, and hence totally dominates every color other than its own. Any other vertex totally dominates the color class of c, so any proper coloring is a td-coloring. Our result follows.

□

Of much interest to us is the upper bound for paths given by Lemma 2.7.

Corollary 2.10. *[2, pg. 958] For $n \geq 2$,*

$$\max\{2, \gamma_{\mathrm{td}}(P_n)\} \leq \chi_{\mathrm{td}}(P_n) \leq \gamma_{\mathrm{td}}(P_n) + 2.$$

Or more specifically,

$$\max\{2, \tfrac{n}{2}\} \leq \chi_{\mathrm{td}}(P_n) \leq \tfrac{n}{2} + 2, \textit{ if } n \equiv 0 \pmod{4}$$
$$\max\{2, \tfrac{n+2}{2}\} \leq \chi_{\mathrm{td}}(P_n) \leq \tfrac{n+2}{2} + 2, \textit{ if } n \equiv 2 \pmod{4}$$
$$\max\{2, \tfrac{n+1}{2}\} \leq \chi_{\mathrm{td}}(P_n) \leq \tfrac{n+1}{2} + 2, \textit{ if } n \equiv 1 \textit{ or } 3 \pmod{4}$$

□

In particular this shows that $\chi_{\mathrm{td}}(P_n) \in \{\gamma_{\mathrm{td}}(P_n), \gamma_{\mathrm{td}}(P_n)+1, \gamma_{\mathrm{td}}(P_n)+2\}$ for all $n \geq 2$.

3. Paths and the case of $n = 18$

Both Henning [2] and Vijayalekshmi [6] give the following values of $\chi_{\mathrm{td}}(P_n)$ for small n. We will state these without proof because they are easy, but quite tedious to check.

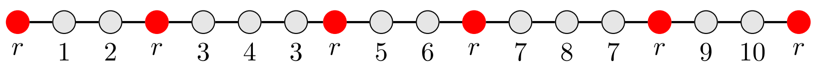
Lemma 3.1. *[2, pg. 972] For $2 \leq n \leq 15$,*

$$\chi_{\mathrm{td}}(P_n) = \begin{cases} \gamma_{\mathrm{td}}(P_n) & \text{for } n \in \{2,3,6\}, \\ \gamma_{\mathrm{td}}(P_n)+1 & \text{for } n \in \{4,5,7,9,10,11,14\}, \\ \gamma_{\mathrm{td}}(P_n)+2 & \text{for } n \in \{8,12,13,15\}. \end{cases}$$

For a base case of one of our main results, we will use $n = 22$ and we will need to know that

$$\chi_{\mathrm{td}}(P_{22}) = \gamma_{\mathrm{td}}(P_{22}) + 2 = 14.$$

Henning used $n = 18$ in his base case, because he assumed that $\chi_{\mathrm{td}}(P_{18}) = \gamma_{\mathrm{td}}(P_{18}) + 2 = 12$. However, this value for $\chi_{\mathrm{td}}(P_{18})$ is incorrect, since the following td-coloring with 11 colors is a counterexample.



In this figure, the label below a vertex is the color of that vertex. The color r (shown in red) is called a *non-dominated* color class, because no vertex totally dominates that color class. In this example, there is only one non-dominated color.

A *non-repeated* color is a color whose color class is a singleton. Or, in other words, a vertex has a non-repeated color if and only if it is the only vertex with that color. In the example above, colors 1, 2, 4, 5, 6, 8, 9, 10 are the only non-repeated colors.

The next result shows that $\chi_{\mathrm{td}}(P_{18}) = 11$, which completes our consideration of this counterexample to [2, prop. 20].

Proposition 3.2. *For the path P_{18} we have $\chi_{\mathrm{td}}(P_{18}) = \gamma_{\mathrm{td}}(P_{18}) + 1 = 11$.*

Proof. The td-coloring shown above also shows that

$$\chi_{\mathrm{td}}(P_{18}) \leq \gamma_{\mathrm{td}}(P_{18}) + 1 = 11.$$

Suppose we had a td-coloring of P_{18} with 10 colors. Since vertices 1 and 18 must totally dominate some color, vertices 2 and 17 each get non-repeated colors. Now if 3 had a repeated color, then the restriction of the coloring to the subgraph induced by $[4, 18] = \{4, 5, 6, \ldots, 18\}$ would be a td-coloring with at most 9 colors, but by Lemma 3.1, we have that $\chi_{\mathrm{td}}(P_{15}) = \gamma_{\mathrm{td}}(P_{15}) + 2 = 10$. So vertex 3, and similarly vertex 16, must have non-repeated colors. So this means the coloring restricted to the induced subgraph on $[2, 17]$ would be a td-coloring. Suppose vertices 1 and 18 have different colors. Since selecting one vertex from each color class of a td-coloring gives us a td-set, selecting $\{1, 18, 2, 17, 3, 16\}$ and then one more vertex from each of the rest of the color classes gives us a td-set with 10 elements, since there are 10 colors. But since 2, 3, 16, 17 are all totally dominated and also totally dominate 1 and 18, removing 1 and 18 from this total dominating set leaves us with another total dominating set, with 8 elements. But $\gamma_{\mathrm{td}}(P_{18}) = 10$. So vertices 1 and 18 must be the same color; call it r. This means $[4, 15]$ is colored with at most 6 colors, including the repeated r. However, the sets $\{4, 6\}$, $\{5, 7\}$, $\{8, 10\}$, $\{9, 11\}$, $\{12, 14\}$,$\{13, 15\}$ are each neighborhoods of some vertex, so they must contain a color class. This cannot be color r, since vertices 1 and 18 are colored r. Therefore $\chi_{\mathrm{td}}(P_{18}) = \gamma_{\mathrm{td}}(P_{18}) + 1 = 11$. □

Replacing P_{18} by P_{22} in our base case will allow us to find the complete list of values for $\chi_{\mathrm{td}}(P_n)$ for all $n \geq 2$, once we find such minimal td-colorings for $n = 8, 13, 15, 22$, so we need the following result.

Proposition 3.3. *[6, pg. 92] For the path P_{22} we have $\chi_{\text{td}}(P_{22}) = \gamma_{\text{td}}(P_{22}) + 2 = 14$.*

Proof. By Corollary 2.10 and Lemma 2.6 it suffices to show $\chi_{\text{td}}(P_{22}) \geq 14$. Suppose $\chi_{\text{td}}(P_{22}) \leq 13$. Just like the argument for P_{18} above, we can say the following.

(i) Vertices 2,3, 20 and 21 all have non-repeated colors.

(ii) Vertices 1 and 22 have the same color, say r.

If 7 is given a repeated color, then since $\{4, 6\}$ and $\{5, 7\}$ must contain a color class, $[1, 7]$ must contain at least 4 color classes. But 7 having a repeated color also means the coloring restricted to $[8, 22]$ is a td-coloring of P_{15} with at, most 13-4=9 colors, contradicting Lemma 3.1.

(iii) Vertex 7 has a non-repeated color.

From (i) and (ii), we get that the coloring restricted to [4,19] uses at most 9 colors, which includes r. Since each of the sets $\{4, 6\}$, $\{5, 7\}$, $\{8, 10\}$, $\{9, 11\}$, $\{12, 14\}$,$\{13, 15\}$ $\{16, 18\}$, and $\{17, 19\}$, contains a color class, if any of these was not a color class, then one vertex in it must be a non-repeated color and the other must be color r. This means

(iv) Vertex 5 has color r, which in turn implies (v).

(v) The set $\{4, 6\}$ is a color class.

This also tells us that 7 must totally dominate 8, which implies (vi).

(vi) Vertex 8 has a non-repeated color.

By symmetry, we have (vii) - (ix).

(vii) Vertices 21, 20, 16, and 15 have non-repeated colors.

(viii) The set $\{17, 19\}$ is a color class.

(ix) Vertex 18 is colored r.

Putting this all together, we have that the coloring restricted to $[9, 14]$ must be colored with at most 3 colors including r. But $\{9, 11\}$, $\{10, 12\}$ must contain color classes, so 13 and 14 must both be colored r, contradicting that this is a proper coloring. □

Vijayalekshmi [6] shows that $\chi_{\text{td}}(P_n) = \gamma_{\text{td}}(P_n) + 2$ for $n = 8, 13$, and 15, but the argument is the same as for 22, only easier, so we omit it here.

Lemma 3.4. *[6, pg. 91] For $n \geq 2$, $\chi_{\text{td}}(P_{n+4}) \geq \chi_{\text{td}}(P_n) + 2$*

Proof. The base cases of $n < 6$ can be seen from Lemma 3.1. Now take a minimal td-coloring of P_{n+4}. If $n + 1$ has a repeated color, then the coloring restricted to the induced subgraph on the vertices $[n]$, is a td-coloring, and $n + 1$, $n + 2$, $n + 3$, and $n + 4$ must contain two color classes since only $n + 3$ is totally dominated by $n + 4$ and $\{n + 2, n + 4\}$ contains a color class. So our desired result follows in this case. Similarly the result follows if vertex 4 is repeated, so we will assume 4 and $n + 1$ are not repeated. But then the coloring restricted to $[3, n + 2]$ is a td-coloring, and since 2 and $n + 3$ have non-repeated colors, we again have $\chi_{\text{td}}(P_{n+4}) \geq \chi_{\text{td}}(P_n) + 2$. □

Corollary 3.5. *[6, pg. 91] If for some n, $\chi_{\text{td}}(P_n) = \gamma_{\text{td}}(P_n) + 2$, then $\chi_{\text{td}}(P_m) = \gamma_{\text{td}}(P_m) + 2$, for all $m > n$ such that $m \equiv n \pmod 4$.*

Proof. By Lemma 3.4 $\chi_{\text{td}}(P_{n+4}) \geq \chi_{\text{td}}(P_n) + 2 = \gamma_{\text{td}}(P_n) + 2 + 2$. By Lemma 2.6, $\gamma_{\text{td}}(P_n) + 2 = \gamma_{\text{td}}(P_{n+4})$, so $\chi_{\text{td}}(P_{n+4}) = \gamma_{\text{td}}(P_{n+4}) + 2$, and our result follows by Corollary 2.10 and induction.

□

We will use this fact later, and we note the following consequence.

Theorem 3.6. *For $n \geq 19$,*

$$\chi_{\rm td}(P_n) = \begin{cases} \frac{n}{2} + 2 & \textit{if} \quad n \equiv 0 \pmod 4, \\ \frac{n+2}{2} + 2 & \textit{if} \quad n \equiv 2 \pmod 4, \\ \frac{n+1}{2} + 2 & \textit{otherwise.} \end{cases}$$

Alternatively, this can also be written as,

$$\chi_{\rm td}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor + 2.$$

Lemma 3.1, Theorem 3.6, and Proposition 3.2, taken together, give us the values of $\chi_{\rm td}(P_n)$ for all $n \geq 2$.

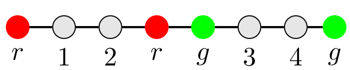
4. Cycles

We need a few more results about paths before moving on to cycles.

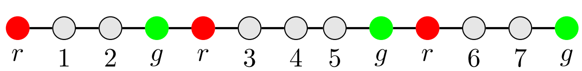
Theorem 4.1. *[6, pg. 94] For $n \geq 2$, if P_n has a minimal td-coloring with at least two non-dominated color classes, then P_{n+4} has a minimal td-coloring with at least two non-dominated color classes.*

Proof. Let the color classes of such a minimal td-coloring of P_n be $C_1, C_2 \ldots C_m$. And let C_1 and C_2 be non-dominated color classes. Then considering P_n as a subgraph of P_{n+4}, we take the same color classes but with C_1 replaced by $C_1 \cup \{n+1\}$, C_2 replaced by $C_2 \cup \{n+4\}$, and the two new color classes $\{n+2\}$ and $\{n+3\}$. This is clearly still a total dominator coloring, and meets the bound of Corollary 3.4 so it is minimal. □

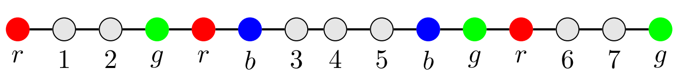
Shown below are the minimal td-colorings with two non-dominated colors, of our base cases.



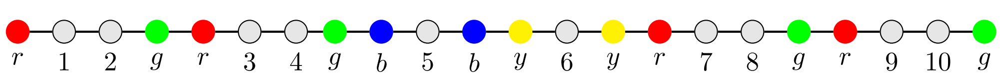
n=8



n=13



n=15



n=22

From these colorings, Corollary 3.5, and Theorem 4.1, we have the following results.

Corollary 4.2. *[6, pg. 94] For all $n \geq 8$, if $\chi_{\rm td}(P_n) = \gamma_{\rm td}(P_n) + 2$, then P_n has a minimal td-coloring with end vertices that are a different color.* □

Corollary 4.3. *[6, pg. 95] For all $n \geq 8$, if $\chi_{\mathrm{td}}(P_n) = \gamma_{\mathrm{td}}(P_n)+2$, then P_n has a minimal td-coloring with at least two non-dominated color classes* □

If there is a minimum td-coloring of P_n where the end vertices have different colors we can connect the ends to get a td-coloring of C_n. So, by the corollaries above, we have the following result.

Lemma 4.4. *For $n \geq 8$, if $\chi_{\mathrm{td}}(P_n) = \gamma_{\mathrm{td}}(P_n) + 2$, then $\chi_{\mathrm{td}}(C_n) \leq \chi_{\mathrm{td}}(P_n)$.* □

Now we will present a lemma and proof from [4].

Lemma 4.5. *[6, pg. 93] For $n \geq 5$, if C_n has a minimal td-coloring in which there exists a color class of the form $N(x)$, (i.e. that is a neighborhood of a vertex), where x has a non-repeated color, or there is no color class of the form $N(x)$, then $\chi_{\mathrm{td}}(P_n) \leq \chi_{\mathrm{td}}(C_n)$.*

Proof. First suppose C_n has a minimal td-coloring in which there exists a color class of the form $N(x)$, and x has a non-repeated color. Without loss of generality, assume x is vertex 2, with non-repeated color c_1, and $N(x) = \{1, 3\}$ is the color class with color c_2. Vertex 1 has a repeated color so n must totally dominate vertex $n-1$, since $n \geq 5$. So our coloring restricted to the path from vertex 1 to n is a td-coloring, so $\chi_{\mathrm{td}}(P_n) \leq \chi_{\mathrm{td}}(C_n)$ in this case.

Now suppose C_n has a minimal td-coloring where there is no color class of the form $N(x)$. This implies that for any vertex with non-repeated color, it must have a neighbor with a non-repeated color. [In fact every vertex x has a neighbor with a non-repeated color since, otherwise, it must totally dominate one of those color classes and hence that color class must be $N(x)$, contradicting our assumption.]
There are three subcases in this case.
Subcase 1 There are no repeated colors. In this case, every vertex totally dominates each of its neighbors. So removing an edge uv, u still totally dominates its other neighbor u_1, and similarly v for its other neighbor v_1. So this is a td-coloring of P_n.
Subcase 2 There are two adjacent vertices u,v with repeated colors. Then the vertex on the other side of u, u_1, and on the other side of v, v_1, must have non-repeated colors. So removal of the edge uv would give a td-coloring of P_n.
Subcase 3 The final subcase is that there are no two adjacent vertices that are both repeated and there are adjacent vertices u, v with one of them having a repeated color and the other having non repeated color. Without loss of generality say u is repeated and v is non-repeated. Again call the vertex on the other side of u, u_1, and on the other side of v, v_1. Vertex u_1 can't be repeated by assumption of this subcase, and v_1 is non-repeated since it must be totally dominated by v. Again removing edge uv will give a td-coloring of P_n.

So in all these cases we have our result $\chi_{\mathrm{td}}(P_n) \leq \chi_{\mathrm{td}}(C_n)$.

□

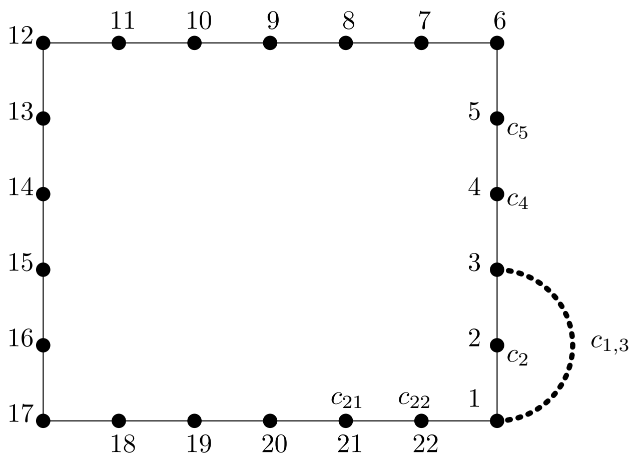
Vijayalekshmi makes the following observations.

$$\chi_{\mathrm{td}}(C_n) = \begin{cases} \chi_{\mathrm{td}}(P_n) - 1 & \text{for } n = 4, \\ \chi_{\mathrm{td}}(P_n) & \text{for } n \in \{5,6,7,8,9,10,12,13,14,15,16,17\}, \\ \chi_{\mathrm{td}}(P_n) + 1 & \text{for } n \in \{3,11,18\}. \end{cases}$$

We will show, just as Vijayalekshmi does in [4], that for all $n \geq 19$, $\chi_{\mathrm{td}}(C_n) = \chi_{\mathrm{td}}(P_n)$. More specifically, we will show that if $\chi_{\mathrm{td}}(P_n) = \gamma_{\mathrm{td}}(P_n) + 2$, then $\chi_{\mathrm{td}}(C_m) = \chi_{\mathrm{td}}(P_m)$ for all $m > n$ and $m \equiv n \pmod 4$, which will give us the values of all remaining $\chi_{\mathrm{td}}(C_m)$ not in the values given above. First we need one more base case.

Theorem 4.6. *[4, pg. 94]* $\chi_{\mathrm{td}}(C_{22}) = \chi_{\mathrm{td}}(P_{22}) = 14$

Proof. By Lemma 4.4 we have $\chi_{\mathrm{td}}(C_{22}) \leq \chi_{\mathrm{td}}(P_{22})$. Suppose $\chi_{\mathrm{td}}(C_{22}) < \chi_{\mathrm{td}}(P_{22}) = 14$. Then by the contrapositve of Lemma 4.5, there is a minimal td-coloring in which there is a color class of the form $N(x)$ where x has a repeated color, say color c_2. Without loss of generality, say x is vertex 2. So vertices 1 and 3 are the same color, say $c_{1,3}$. First we will assume the color class of c_2 is not $N(1)$ or $N(3)$. In this case we can see that since $c_{1,3}$ is repeated, vertex 4 and 22 have non-repeated colors, say c_4 and c_{22} respectively. Since $c_{1,3}$ is repeated, 5 and 21 must also get non repeated colors, say c_5 and c_{21}.



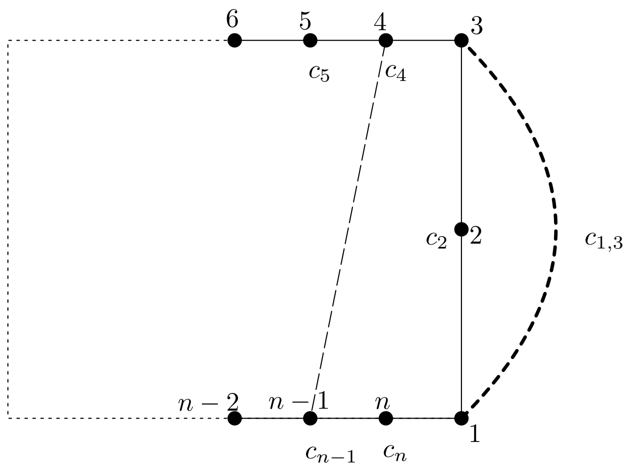
The coloring restricted to the induced subgraph on the vertices $[6, 20]$, is a proper coloring with 8 colors, including c_2 (it is not necessarily a td-coloring). The seven sets $\{6, 8\}$, $\{7, 9\}$, $\{10, 12\}$, $\{11, 13\}$, $\{14, 16\}$, $\{15, 17\}$, and $\{18, 20\}$, must each contain a color class, since they are neighbor sets, so 19 must get color c_2, and by symmetry 7 also gets color c_2. This in turn means 9 and 17 must have non-repeated colors, and that $\{6, 8\}$ and $\{18, 20\}$ are color classes. This means that 10 and 16 have non-repeated colors. So we have that the vertices [11,15] are colored with 2 colors including c_2, but that is not possible since $\{11, 13\}$ and $\{12, 14\}$, must contain color classes.

The case where the color class of vertex 2 is $N(1)$ or, $N(3)$, without loss of generality say $N(3)$. Call its color $c_{2,4}$, and notice it can't be used elsewhere in the graph since vertex 3 must totally dominate it. Similar to the argument before, we can see that vertices 5, 6, 22, and 21 must have non-repeated colors. So now the vertices $[7, 20]$, are colored with 7 colors, none of which can be the colors used to color vertices $[21, 6]$. However that means the sets $\{7, 9\}$, $\{8, 10\}$, $\{11, 13\}$, $\{12, 14\}$, $\{15, 17\}$, and $\{16, 18\}$, must be color classes. But that means vertices 19 and 20 get the same color, which is impossible. Therefore we have $\chi_{\mathrm{td}}(C_{22}) = \chi_{\mathrm{td}}(P_{22}) = 14$. □

Theorem 4.7. *[4, pg. 95] For $n \geq 5$, If $\chi_{\mathrm{td}}(P_n) = \gamma_{\mathrm{td}}(P_n) + 2$, then $\chi_{\mathrm{td}}(C_m) = \chi_{\mathrm{td}}(P_m)$, for all $m > n$ such that $m \equiv n \pmod 4$*

Proof. By Lemmas 4.4 and 4.5 and the values we have found, it suffices to show that $\chi_{\mathrm{td}}(P_m) \leq \chi_{\mathrm{td}}(C_m)$ when there is a color class of the form $N(x)$ where x is repeated. We will call the vertex x, vertex 2. We proceed by induction on n. The base cases of interest are when $n = 8, 13, 15$, and 22 which are routine to check. Now suppose the result is true for $n - 4$. Say vertex 2 has color c_2, and vertices 1 and 3 have color $c_{1,3}$. Since these are repeated colors, vertices 4, 5, n, and

$n-1$ must have non-repeated colors. We can remove vertices $n-1$, 1, 2, and 3, and add the edge $\{4, n-1\}$, and this will give us a td-coloring of C_{n-4}. So by our inductive hypothesis, we have $\chi_{\mathrm{td}}(C_n) \geq \chi_{\mathrm{td}}(C_{n-4}) + 2 = \chi_{\mathrm{td}}(P_{n-4}) + 2 = \chi_{\mathrm{td}}(P_n)$, where the last equality is by Corollary 3.5.



The dotted curve on the right in the figure above represents that vertex 1 and 3 are both colored $c_{1,3}$. The dashed edge from 4 to $n-1$ is the edge that we add after removing vertices n, 1, 2, 3. □

This completes our list of values for $\chi_{\mathrm{td}}(P_n)$ for all $n \geq 2$ and $\chi_{\mathrm{td}}(C_n)$ for all $n \geq 3$.

References

[1] C. D. Godsil and G. Royle. *Algebraic Graph Theory*. Springer-Verlag, New York, 2001.

[2] Michael A. Henning. Total dominator colorings and total domination in graphs. *Graphs and Combinatorics*, 31:953–974, 2015.

[3] Adel P. Kazemi. Total dominator chromatic number of a graph. *Transactions on Combinatorics*, 4 No. 2:57–68, 2015.

[4] A. Vijayalekshmi. Total dominator colorings in cycles. *International J. Math. Combinatorics*, 4:92–96, 2012.

[5] A. Vijayalekshmi. Total dominator colorings in graphs. *International Journal of Advancement in Research & Technology*, 1, 2012.

[6] A. Vijayalekshmi. Total dominator colorings in paths. *International J. Math. Combinatorics*, 2:89–95, 2012.